

Hoop Conjecture and Black Holes on a Brane

Kouji NAKAMURA^{1,*}), Ken-ichi NAKAO^{2,**}) and Takashi MISHIMA^{3,***})

¹*Division of Theoretical Astrophysics, National Astronomical Observatory,
Mitaka 181-8588, Japan*

²*Department of Physics, Graduate School of Science, Osaka City University,
Osaka 558-8585, Japan*

³*Laboratories of Physics, College of Science and Technology, Nihon University,
Funabashi 274-0063, Japan*

(Received April 30, 2002)

The initial data of gravity for a cylindrical matter distribution confined to a brane are studied in the framework of the single-brane Randall-Sundrum scenario. In this scenario, the 5-dimensional nature of gravity appears in the short-range gravitational interaction. We find that a sufficiently thin configuration of matter leads to the formation of a marginal surface, even if the configuration is infinitely long. This implies that the hoop conjecture proposed by Thorne does not hold on the brane: Even if a mass M does not become compacted into a region whose circumference \mathcal{C} in every direction satisfies $\mathcal{C} > 4\pi GM$, black holes with horizons can form in the Randall-Sundrum scenario.

§1. Introduction

Black hole horizons form when and only when a mass M gets compacted into a region whose circumference \mathcal{C} in every direction is $\mathcal{C} < 4\pi GM$. This is the statement of Thorn's hoop conjecture for the necessary and sufficient condition for black hole formation in 4-dimensional general relativity.¹⁾ Assuming physically reasonable conditions on the matter fields, Thorne has proven that there is no marginal surface in a system with a cylindrical distribution of matter fields.¹⁾ In his proof, the marginal surface is a cylindrically symmetric space-like 2-surface such that the expansion of the outgoing null normal to this surface vanishes. This result, together with the Newtonian analogy, led to his conjecture.

Because of the ambiguities in the above statement, there are many proposals for a precise reformulation and attempts to prove this conjecture.^{3)–5)} Among them, the numerical simulations of Nakamura et al.³⁾ and Shapiro and Teukolsky⁴⁾ suggest that the hoop conjecture holds in 4-dimensional general relativity. Therefore, we may assume that this conjecture gives one of the criteria for black hole formation and a highly elongated matter distribution does not form a black holes.

However, strictly speaking, we do not know whether general relativity can describe strong gravity in our universe even for classical situations? There is no experimental evidence for it. If general relativity is inapplicable to the situation of the strong gravity, it again becomes a non-trivial issue whether or not the hoop

*) E-mail: kouchan@th.nao.ac.jp

**) E-mail: knakao@sci.osaka-cu.ac.jp

***) E-mail: tmishima@ns.phys.ge.cst.nihon-u.ac.jp

conjecture gives a reasonable criterion of black hole formation.

Brane worlds are regarded as alternative theories of gravity. Among several models, one of the simplest is the single-brane model proposed by Randall and Sundrum (RS II model).⁶⁾ One of the interesting features of this model is the fact that it reproduces 4-dimensional Newtonian and general relativistic gravity on the brane to more than adequate precision. The deviation in the gravitational force from the Newtonian one appears at short length scales less than $l := \sqrt{-6/\Lambda} = 3/(4\pi G_5 \lambda)$, where $\Lambda < 0$ is the negative cosmological constant and $\lambda > 0$ is the positive tension of the brane. The length scale of l must be less than the order of one millimeter due to the experimental constraints.⁷⁾

It is natural to expect that 5-dimensional aspects of the RS-scenario will appear in the short-range force of gravity (the scale less than l), and hence in strong gravity with spacetime curvature radius less than l . Further, we can expect that the hoop conjecture applied to the RS brane may not be valid on scales less than l for the following reason. In the RS-scenario, the cancellation of the long-range forces due to the negative cosmological constant and the brane tension results in the reproduction of the Minkowski spacetime, and 4-dimensional gravity is reproduced by the warp factor, due to the negative cosmological constant in the bulk. Note that both effects are due to the cosmological constant on the brane and bulk. We should note that short-range gravity is not particularly sensitive to the cosmological constant. The 5-dimensional aspects of the RS II model will appear on short-range scales. Since there are so-called “black string solutions” in 5-dimensional Einstein gravity. These are cylindrically symmetric black holes in 5-dimensional spacetime. Thus a highly elongated matter distribution may form due to the effect of 5-dimensional gravity.

In this article, we confirm the above conjectures assuming the existence of an infinite energy density of the brane. For simplicity, we concentrate on the time symmetric initial data, which is a 4-dimensional space-like hypersurface embedded in the entire spacetime with vanishing extrinsic curvature.⁸⁾ Further, we consider a cylindrically symmetric matter distribution on the brane and the formation of the cylindrical marginal surface on the brane, since there is no marginal surface in this system within 4-dimensional general relativity. We show that a cylindrical marginal surface does form in the RS II model. Next, we discuss the hoop conjecture on the RS-brane.

§2. Time symmetric initial value constraint

In the RS II model, the entire spacetime is governed by 5-dimensional Einstein gravity. The geometry of the 4-dimensional time-symmetric initial hypersurface (Σ, q_{ab}) embedded in the 5-dimensional spacetime should satisfy the Hamiltonian constraint

$${}^{(4)}R = 16\pi G_5 T_{\perp\perp}, \quad (2.1)$$

where ${}^{(4)}R$ is the scalar curvature on Σ , $T_{\perp\perp} := T_{ab}u^a u^b$, T_{ab} is the 5-dimensional energy momentum tensor, and u^a is the time-like unit vector normal to Σ . In the

context of the RS II model, $T_{\perp\perp}$ is given by

$$T_{\perp\perp} = \frac{\Lambda}{8\pi G_5} + (\lambda + \rho) \delta(\chi), \quad (2.2)$$

where ρ is the energy density of the matter field confined to the brane, and χ is the Gaussian normal coordinate of the brane.

The existence of the brane on the initial surface gives a discontinuity of the extrinsic curvature defined by $\kappa_{ab} := -h_a^c h_b^d D_c n_d$, where n_a is the unit vector normal to \mathcal{S} , $h_{ab} := q_{ab} - n_a n_b$, and D_a is the covariant derivative associated with the metric q_{ab} . This discontinuity is derived from the initial value constraint (2.1). The scalar curvature on Σ is given by

$${}^{(4)}R = {}^{(3)}R - (\kappa^a_a)^2 - \kappa_d^c \kappa_c^d + 2 \frac{\partial}{\partial \chi} \kappa_a^a, \quad (2.3)$$

where χ is the Gaussian normal coordinate of \mathcal{S} . Then, the discontinuity of κ^a_b with the Z_2 -symmetry at the brane is given by

$$\kappa^a_a = 4\pi G_5 (\lambda + \rho). \quad (2.4)$$

We assume that the intrinsic geometry on the initial data is described by the following line element:

$$d\ell^2 := \phi^2 d\tilde{\ell}^2 = \phi^2 (dR^2 + R^2 d\varphi^2 + dz^2 + d\psi^2). \quad (2.5)$$

Here, ψ is the coordinate for the extra dimension, which is chosen so that the brane is at $\psi = l$ on this initial data and the brane normal is given by $n_a = (d\chi)_a = \phi(d\psi)_a$. (R, φ, z) represents the spatial cylindrical coordinate system on the 3-dimensional subspace orthogonal to n_a . The conformal factor ϕ is determined by the initial value constraint (2.1) and the boundary condition at the brane (2.4). These are given by

$$\tilde{D}^a \tilde{D}_a \phi + \frac{1}{3} \phi^3 \Lambda = 0, \quad (2.6)$$

$$\partial_\psi \phi + \frac{4\pi G_5}{3} (\lambda + \rho) \phi^2 \Big|_{\psi=l} = 0, \quad (2.7)$$

where \tilde{D}_a is the covariant derivative associated with the flat line element $d\tilde{\ell}^2$. Using the fine tuning made by choosing $l := \sqrt{-6/\Lambda} = 3/(4\pi G_5 \lambda)$ ($\rho = 0$), the time-symmetric hypersurface of the Minkowski brane in the RS II model is given by $\phi = l/\psi$. Since we concentrate on the cylindrically symmetric matter distribution on the brane, we solve Eq. (2.6) under the boundary condition (2.7) with $\rho = \rho(R)$.

§3. Cylindrical marginal surfaces on the brane

As explained in Ref.9), we numerically solved (2.6). To solve (2.6), we consider the energy density

$$\rho(R) \phi^2|_{\psi=l} = \frac{3\sigma}{\pi R_s^2} \left\{ \left(\frac{R}{R_s} \right)^2 - 1 \right\}^2 \quad (3.1)$$

on the brane. Further, we imposed boundary conditions as follows: (i) the junction condition (2.7) at the brane; (ii) regularity at the axis of the cylindrical radial coordinate $R = 0$ by $\partial_R \phi = 0$; (iii) the condition that $\phi - l/\psi$ behaves like the linear solution $\Omega_L = O(G_5 l \rho) \ll 1$ with a singular line source $\rho(R) = \sigma_L \delta(R)/2\pi R$ near the numerical boundaries $\psi = \psi_{\max}$ and $R = R_{\max}$.

The boundary condition (iii) for the conformal factor ϕ guarantees that our cylindrical matter distribution is an isolated system. The linear solution of Eq.(2.6) with (2.7) is given by

$$\Omega_L := \frac{2G_5 \sigma_L}{3} \left\{ \frac{3l}{\psi^2} \ln \left(\frac{R_c}{R} \right) + \int_0^\infty dm u_m(l) u_m(\psi) K_0(mR) \right\}, \quad (3.2)$$

where R_c is an integration constant, $K_0(x)$ is the modified Bessel function of the 0th kind, and $u_m(\psi)$ is a combination of the spherical Bessel functions of the first and second kinds:

$$u_m(\psi) = \sqrt{\frac{2(ml)^4}{\pi((ml)^2 + 1)}} m \psi (n_1(ml) j_2(m\psi) - j_1(ml) n_2(m\psi)). \quad (3.3)$$

Though the mode function (3.3) is slightly different from that for the static perturbation,⁶⁾ the linear solution (3.2) on the brane is reproduced to more than adequate precision on large length scales. Actually, the asymptotic form of Ω_L on the brane is given by

$$\Omega_L \sim \frac{2G_5 \sigma_L}{l} \left(\ln \frac{R_c}{R} + \frac{l^3}{3R^3} \right). \quad (3.4)$$

The logarithmic behavior of this asymptotic form is due to the zero mode. This behavior is interpreted as the asymptotically conical structure of the initial surface, as in 4-dimensional Einstein gravity.

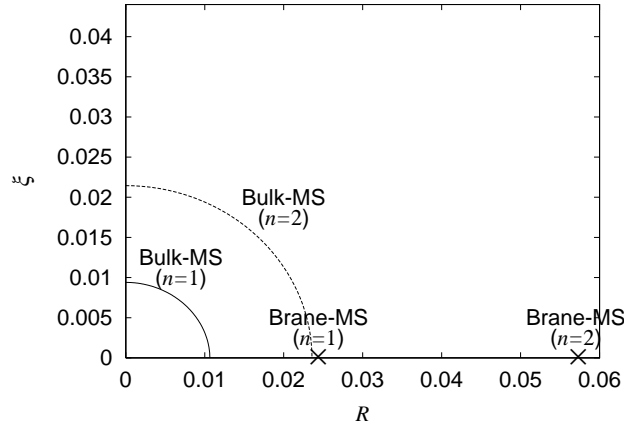


Fig. 1. The Bulk-MS and the Brane-MS are depicted. The matter is located within the region $0 \leq R < 0.01n$ at $\xi = 0$, where $n = 1, 2$.

On the initial surface with the numerical data, we found the cylindrical marginal surfaces. As commented by Shiromizu and Shibata,⁸⁾ there are two conceptually different marginal surfaces on the initial surface: One is defined by the “null” rays confined to the brane (*Brane-MS*) and the other is defined by the null rays that propagate in the entire spacetime, including the bulk (*Bulk-MS*). Brane-MS is the marginal surface for all the physical fields confined to the brane, and Bulk-MS is for all the physical fields including gravitons that propagate in the entire spacetime. Brane-MS does not depend on the causal structure of the entire spacetime, but Bulk-MS does. The marginal surfaces we found are shown in Fig. 1. Thus we found a cylindrical marginal surface on the brane that does not form in 4-dimensional Einstein gravity, as Thorne proved. These cylindrical marginal surfaces can be regarded as counter examples of the hoop conjecture by introducing appropriate definitions of the mass M and the circumference \mathcal{C} in the next section.

§4. Hoop conjecture and infinite cylindrical matter distribution

Because we found a cylindrical marginal surface in the RSII model, we can say that the hoop conjecture is essentially violated in this model, i.e., a black hole may form even if the matter distribution is highly elongated. Further, the above examples are also regarded as counter-examples of the hoop conjecture. To see this, we must introduce definitions of the mass M and the circumference \mathcal{C} in the hoop conjecture and then we consider the marginal surface in the situation $\mathcal{C} > 4\pi GM$.

As the definition of the mass, we adopt the total proper mass

$$M := 2\pi \int_0^{R_s} dR R \int_{-L}^L dz \phi^3(R, 0) \rho(R) \quad (4.1)$$

within a cylinder of coordinate radius R_s and finite coordinate length $2L$. As the definition of the circumference, we adopt the proper length

$$\mathcal{C} := 2 \int_{-L}^L dz \phi(R_s, 0) + 4 \int_0^{R_s} dR \phi(R, 0) \quad (4.2)$$

of this cylinder. We can easily check that the inequality $\mathcal{C} > 4\pi GM$ holds for arbitrary L if and only if the following inequality is satisfied:

$$\frac{4\pi^2 G_5}{l\phi(R_s, 0)} \int_0^{R_s} dR R \phi^3(R, 0) \rho(R) \leq 1. \quad (4.3)$$

Since the inequality (4.3) is independent of the coordinate length L , we can apply this inequality to an infinite cylinder. This inequality gives an upper bound on the line energy density of the cylindrical matter field.

We also confirmed numerically that the inequality (4.3) holds in the above examples, and these can be regarded as counter-examples of the hoop conjecture. Therefore, we may say that *the Bulk-MS can form even for such a highly elongated matter distribution that the inequality $\mathcal{C} > 4\pi GM$ is satisfied.* In other words, the inequality $\mathcal{C} > 4\pi GM$ is not a necessary condition for the formation of black holes with

horizons in the RS scenario, although it might be a sufficient condition. In spite of the fact that cylindrical marginal surfaces do not exist in 4-dimensional general relativity, our examples suggest that massive spindle singularities in the RS scenario will be enclosed by the marginal surface.

§5. Discussion

In this article, we have discussed the violation of the hoop conjecture in the RS II model. The existence of a cylindrical marginal surface demonstrated in this article is due to the 5-dimensional effect of gravity. Because we impose boundary conditions such that the spatial infinity on the initial surface is flat with a finite deficit angle, we conjecture that the existence of future null infinity similar to that in the AdS or Minkowski spacetime and, further, that the global hyperbolicity in the causal past of the future null infinity are guaranteed. If these conjectures hold, the formation of a marginal surface in the initial data means the formation of a black hole with an event horizon.

Though there will be static solutions with cylindrical event horizons along the brane, we also expect that these cylindrical horizon might be unstable, because many higher-dimensional black string solutions are unstable.¹⁰⁾ However, even though the examples above are unstable, they show that a highly elongated marginal surface may form in the RS model as a transient phase. This is one of the features of the RS model as an alternative theory of gravity.

Though the inequality $\mathcal{C} > 4\pi GM$ is a sufficient condition for the formation of black holes in RS model, as shown above, this inequality gives an upper bound on the radius of the black string on the brane for the following reason. The line energy density σ is roughly estimated as $\sigma \sim M/(\mathcal{C}/2)$, where M is the mass of a portion whose circumference is \mathcal{C} . If the inequality $\mathcal{C} < 4\pi GM$ holds, black holes will form in the usual sense of 4-dimensional general relativity. However, when the inequality $\mathcal{C} > 4\pi GM$ holds, the 4-dimensional effect does not produce a black hole but black holes may form due to the 5-dimensional effects. The inequality $\mathcal{C} > 4\pi GM$ suggests $G_5\sigma < l/2\pi \sim 0.16l$ in the RS II model. Because the radius of a 5-dimensional black string is also roughly estimated by $r \sim G_5\sigma$, this inequality suggests that the radius of the black string should be of order $O(0.1l)$ or smaller.

Our examples depicted in Fig. 1 satisfy the upper bound of the black string radius estimated above. In other words, our examples also show that on a scale a sufficiently shorter range than l , black string solutions on the brane will be approximated well by 5-dimensional black string solutions, in spite of the existence of a timelike singular hypersurface (brane). Thus, in our example, the cosmological constant and the tension of the brane do not lead to essential effects on such a scale and, the 5-dimensional features in the gravitational interaction appear as expected. To clarify whether or not the above rough estimation is reasonable for various black string solutions, it is necessary to consider black string solutions with different thickness of matter distribution and line energy density.

Acknowledgements

We are grateful to D. Ida, A. Ishibashi, H. Ishihara, T. Tanaka, G. Uchida and H. Kodama for their useful comments and discussions. This work was partially supported by a Grant-in-Aid for Creative Basic Research (No. 09NP0801) from the Japanese Ministry of Education, Science, Sports and Culture.

References

- 1) K. S. Thorne, in *Magic Without Magic; John Archibald Wheeler*, ed. J. Klauder (Frieman, San Francisco, 1972), p. 231.
- 2) R. Penrose, Riv. Nuovo Cim. **1** (1969), 252.
- 3) T. Nakamura, K. Maeda, S. Miyama and M. Sasaki, *Proceedings of the 2nd Marcel Grossmann Meeting on General Relativity*, ed. R. Ruffini (North-Holland, Amsterdam, 1982), p. 675.
- 4) S. L. Shapiro and S. A. Teukolsky, Phys. Rev. Lett. **66** (1991), 994.
- 5) T. Nakamura, S. L. Shapiro and S. A. Teukolsky, Phys. Rev. D **38** (1988), 2972.
 J. Wojtkiewicz, Phys. Rev. D **41** (1990), 1867.
 E. Flanagan, Phys. Rev. D **46** (1992), 1429.
 A. M. Abrahams, K. R. Heiderich, S. L. Shapiro and S. A. Teukolsky, Phys. Rev. D **46** (1992), 2452.
 E. Malec, *ibid*, **67** (1991), 949.
 J. Wojtkiewicz, Class. Quantum Grav. **9** (1992), 1255.
 T. P. Tod, Class. Quantum Grav. **9** (1992), 1581.
 T. Chiba, T. Nakamura, K. Nakao and M. Sasaki, Class. Quantum Grav. **11** (1994), 431.
 C. Barrabès, A. Gramain, E. Lesigne and P. S. Letelier, Class. Quantum Grav. **9** (1992), L105.
- 6) L. Randall and R. Sundrum, Phys. Rev. Lett. **83** (1999), 4690.
 J. Garriga and T. Tanaka, Phys. Rev. Lett. **84** (2000), 2778.
 M. Sasaki, T. Shiromizu and K. Maeda, Phys. Rev. D **62** (2000), 024008.
 T. Shiromizu, K. Maeda and M. Sasaki, Phys. Rev. D **62** (2000), 024012.
- 7) C. D. Hoyle, U. Schmidt, B. R. Heckel, E. G. Adelberger, J. H. Gundlach, D. J. Kapner and H. E. Swanson, Phys. Rev. Lett. **86** (2001), 1418.
- 8) T. Shiromizu and M. Shibata, Phys. Rev. D **62** (2000), 7502.
- 9) K. Nakao, K. Nakamura and T. Mishima, gr-qc/0112067.
- 10) R. Gregory and R. Laflamme, Phys. Rev. Lett. **70** (1993), 2837. R. Gregory, Class. Quantum Grav. **17** (2000), L125.